Nonlinear Circuit Analysis – An Introduction

1. Why nonlinear circuits?

Electrical devices (amplifiers, computers) are built from nonlinear components. In order to understand the design of these devices, a fundamental understanding of nonlinear circuits is necessary. Moreover, nonlinear circuits is where the "real engineering" comes in. That is, there are no hard and fast rules to analyze most nonlinear circuits – you have to use your brain! But, to make your life easy we will start with some systematic methods to analyze op-amp nonlinear circuits.

This chapter is organized as follows: first we will talk about what makes a circuit nonlinear. Next, we will see a very useful nonlinear circuit – the negative resistance converter. Then we will see an application of the negative resistance converter: the oscillator.

2. What is a nonlinear circuit?

It is easy to understand the difference between a linear and a nonlinear circuit by looking at the difference between a linear and a nonlinear equation:

$$y = x + 2 \qquad (1)$$

$$y = x^2 \qquad (2)$$

The x-y graph of each function is shown below:

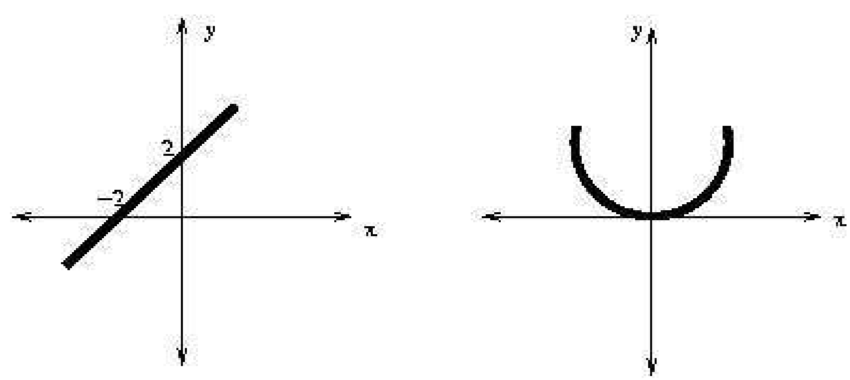


Figure 1. A linear versus nonlinear function

You can see that if the x-y graph of function is a straight line, then obviously the function is linear. But, what about functions like the absolute value:

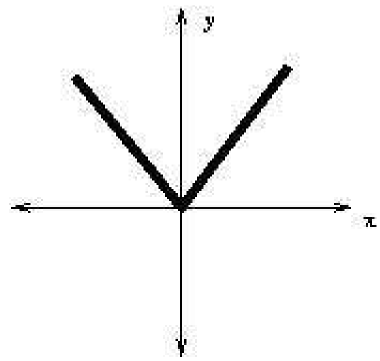


Figure 2. The absolute value function is piece-wise linear

The function above is still classified as nonlinear because we cannot write the function in the form y = ax+b.

In the circuit world, we have i-v graphs. Therefore, **we classify a circuit as linear or non-linear by examining its i-v graph.** If the i-v graph of the circuit is a straight line, then the circuit is classified as linear. Note that the definition can be extended even to circuit elements. For instance, a resistor's i-v graph is a straight line, hence it is a linear device.

3. The Negative Resistance Converter

Consider the op-amp circuit shown below:

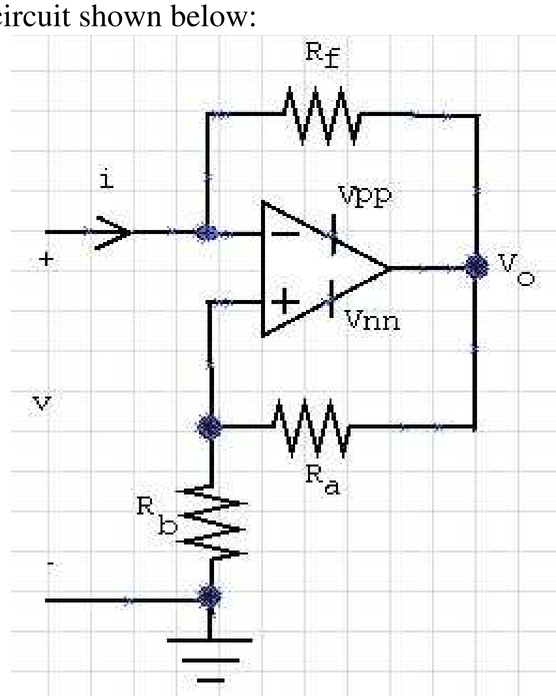


Figure 3. The Negative Resistance Converter

The goal is to derive the i-v graph at the terminals indicated. First, notice this op-amp circuit has both positive and negative feedback. In an op-amp you know that no current flows into the inverting terminal, so we can easily find the relationship between i, V_o and V using Ohm's law:

$$i = \frac{(v - V_o)}{R_f} \tag{3}$$

You should understand from the passive sign convention why the numerator is $v - V_o$, NOT $V_o - v$. Equation (3) above does have an i-v relationship. However, we need to eliminate V_o. Here is where the positive feedback helps us. Again, the current flowing into the non-inverting terminal of the op-amp is zero. The trick is to spot the voltage divider at the non-inverting terminal. Since no current is flowing into the non-inverting terminal, R_a and R_b are in series as shown below:

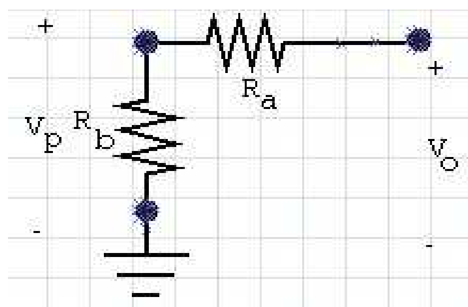


Figure 4. The voltage divider between the output terminal and the non-inverting terminal

V_p (the voltage at the non-inverting terminal) can be easily found using the voltage divider:

$$V_p = \frac{R_b}{(R_a + R_b)} \times V_o \tag{4}$$

$$V_o = \frac{(R_a + R_b)}{R_b} \times V_p \tag{5}$$

But, $V_p = V_n$ (we are assuimg the op-amp is in the linear region). Notice however from figure 3 that $v = V_n$. Substituting in (5):

$$V_o = \frac{(R_a + R_b)}{R_b} \times v \tag{6}$$

Substituting (6) in (3), we get an i-v relationship for the op-amp operating in the linear region:

$$i = \frac{(v - \frac{(R_a + R_b)}{R_b} \times v)}{R_f} \tag{7}$$

Simplifying:

$$i = \frac{-R_a}{(R_b R_f)} \times v \tag{8}$$

The i-v graph for the circuit when the op-amp is operating in the linear region is shown below:

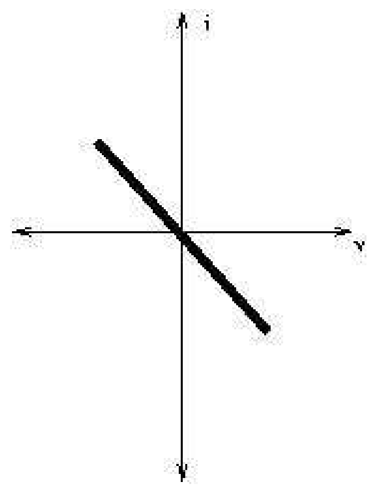


Figure 5. The i-v graph of the circuit when the op-amp is in the linear region.

You can see why the circuit is called a “negative-resistance converter” - the i-v graph is a straight line but unlike a resistor, it has a negative slope. Also notice the circuit is still “linear” - the graph is a straight line. Intuitively, this makes sense. The op-amp is the device in the circuit that causes the nonlinearity. This will occur if the op-amp is saturated. Since we derived the segment in figure 5 by assuming the op-amp is linear, the i-v graph is a straight line.

Let us see what happens when the op-amp saturates. Let us take consider the negative saturation region. That is, V_o = Vnn. Equation (3) is still valid since no current enters the inverting or non-inverting terminal even if the op-amp is saturated:

$$i = \frac{(v - V_{nn})}{R_f} \tag{9}$$

However, to plot the function above we have to figure out for what value of v is the op-amp saturated.

Intuitively, you know any op-amp obeys the op-amp equation in the linear region:

$$\mathbf{V_o} = \mathbf{A}(\mathbf{V_p} - \mathbf{V_n}) \qquad (10)$$

where V_o is the voltage at the ouptut, A is the open loop gain (usually huge like 10^6), V_p is the voltage at the inverting terminal and V_n is the voltage at the noninverting terminal. The op-amp will then rail to Vnn if $V_p < V_n$, in other words if $V_n > V_p$. In our circuit, V_n is v (refer to figure 3). Therefore if $v > V_p$, then the circuit will rail to the Vnn. That is, V_o = Vnn. Then, V_p becomes $V_p = \frac{R_b}{(R_a + R_b)} \times Vnn$ (from equation (4)). Therefore, the point when the op-amp switches from linear to saturation is when $V_n = V_p$ or $v = \frac{R_b}{(R_a + R_b)} \times Vnn$. Now, Vnn is usually negative (like -12 V for the LM741 op-amp). That is why in figure 5 I have extended the graph to the negative v region.

Which way does the i-v graph extend when the op-amp is saturated. If you look at equation (9), you can see that if Vnn is negative then i has to be positive. The reason is: v will always be smaller than Vnn since $v = \frac{R_b}{(R_a + R_b)} \times Vnn$. Therefore, the numerator in equation (9) is always positive. If you stare at figure 3 long enough you will realize that when the op-amp is saturated with V_o = Vnn then V_o is at the lowest possible voltage in the circuit. Therefore current i in figure 3 has to flow from left to right, since v is higher than V_o. This is in the direction of positive current. By symmetry, the argument(s) above apply for the positive saturation and hence we can complete figure 5:

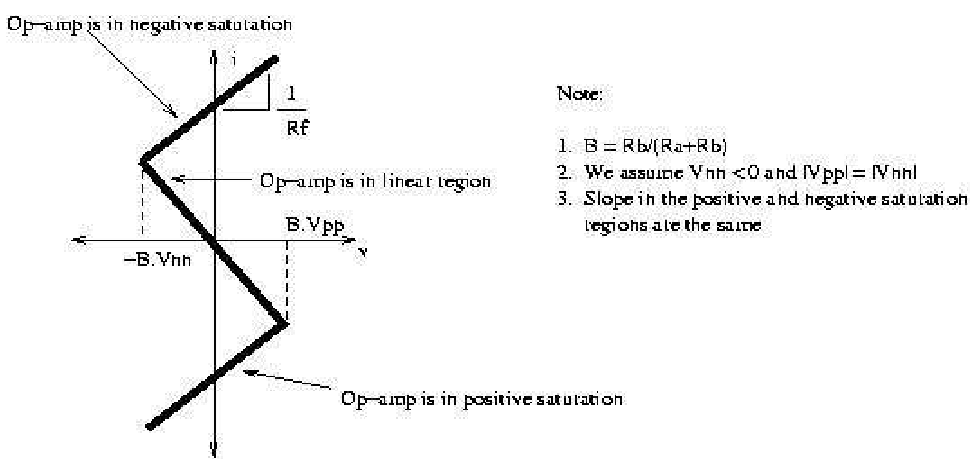


Figure 6. The i-v graph of the negative resistance converter.

That concludes the derivation of the i-v graph for the negative resistance converter. An important observation is the manner in which we derived the graph. We really did not use any "step-by-step" method to derive the result. Rather, we used our intuitive knowledge of how the op-amp works. This is what I call "real-world engineering". But, as I promised you earlier, you can solve a lot of nonlinear problems systematically based on the negative resistance converter. This is the subject of the next section.

4. Oscillators

In this section, we will analyze the negative resistance converter with a capacitor connected across the v terminal. We will assume the capacitor is initially uncharged, i.e.., v(0) = 0 volts. The circuit is shown in figure 7 below.

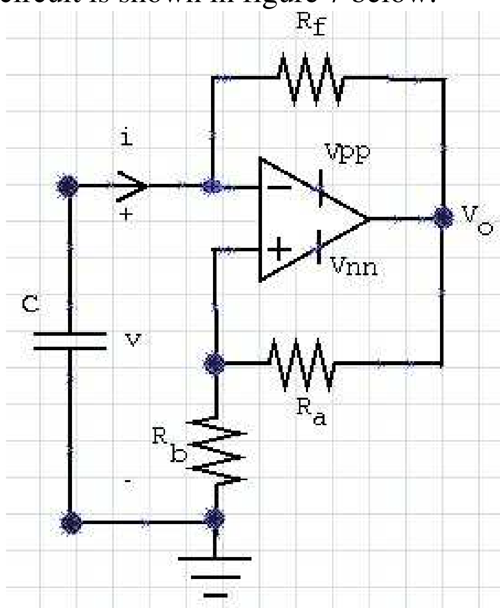


Figure 7. If we attach a capacitor across the input terminals, we will get an oscillatory circuit.

In order to simplify the analysis, we can consider the op-amp as a black box:

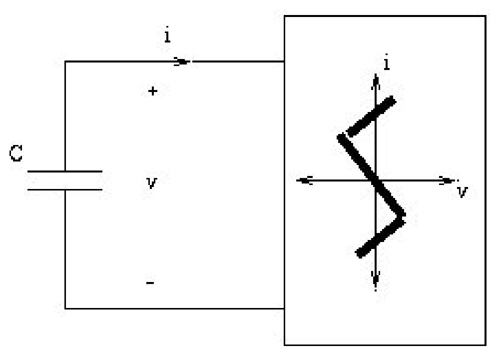


Figure 8. Only the input terminals of the negative resistance converter are important, since we already derived the i-v graph of the circuit.

From figure (8):

$$\mathbf{i} = -\mathbf{C}\left(\frac{\mathbf{dv}}{\mathbf{dt}}\right) \qquad (11)$$

NOTICE THE NEGATIVE SIGN. This is because of the passive sign convention. In figure 8, the current is leaving the positive terminal of the capacitor so we need to have a negative sign in equation (11). If you tried to flip the direction of current to get a positive sign in equation (11), you have to flip the i-v graph for the negative resistance converter! This is because we derived the i-v graph with the sign convention shown in figure 8. It is easier to have a negative sign in equation (11) than flipping an entire i-v graph!

One way to analyze the circuit is to realize that we have 3 straight line regions in the graph: when the op-amp is linear, when the op-amp is in negative saturation and when the op-amp is in positive saturation. Therefore, we can get a linear model for the circuit in each region and do the analysis. However, this is cumbersome and not very intuitive.

Let us analyze the circuit intuitively. First, we need some terminology related to differential equations:

1. Equilibrium Point: $\mathbf{f}'(\mathbf{t}) \equiv 0$

The above expression means an equilibrium point is defined when the derivative of the function is zero. This makes sense physically. Equilibrium is when nothing changes, mathematically, the derivative has to be zero at equilibrium since derivative means rate of change.

Next we will classify equilibrium points as **stable** or **unstable.** It is easier to see the classification with an example. Let us use our negative resistance converter circuit from figure 8. The i-v graph is reproduced below, but now I have added other information:

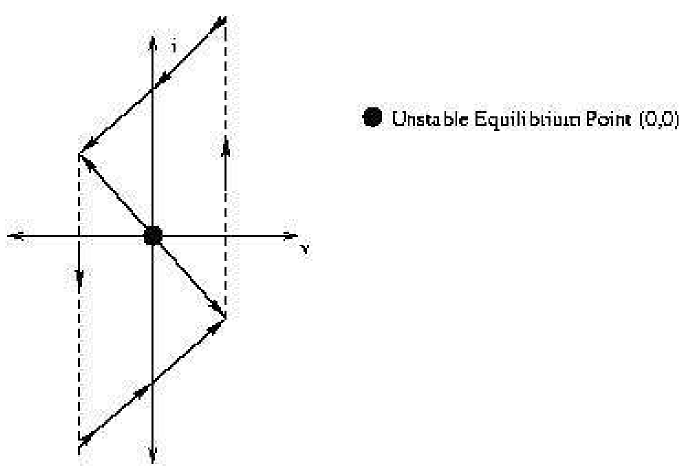


Figure 9. The dynamic route and equilibrium points for the circuit in figure 8.

First, let us find the equilibrium points. If $\mathbf{f'(t)} \equiv 0$ for equilibrium we can use equation (11) since it has a derivative.

$$(\frac{\mathbf{dv}}{\mathbf{dt}}) = 0 \qquad (12)$$

$$\rightarrow \quad \frac{-\mathbf{i}}{\mathbf{C}} = 0 \qquad (13)$$

$$\rightarrow \quad \mathbf{i} = 0 \qquad (14)$$

Equation (14) implies the equilibrium points lie on the x-axis. Figure 9 shows the only equilibrium point for this circuit – (0,0). Notice equation (14) intuitively makes sense. At equilibrium, capacitors are fully charged (or discharged) since all the transients have died out. That is, a capacitor is an open circuit at equilibrium - the current flowing through the capacitor at equilibrium is zero.

2. Stability

Figure (9) has some arrows marked on the i-v graph, these indicate the **dynamic route** of the circuit and helps us obtain the stability of equilibrium points. The dynamic route shows how the voltages and current in a circuit change given an initial condition. Equation (11) again helps us determine the dynamic route. From equation (11);

$$\mathbf{i} > 0 \rightarrow \mathbf{v'} < 0 (\mathbf{C} > 0) \qquad (15)$$

$$\mathbf{i} < 0 \rightarrow \mathbf{v'} > 0 (\mathbf{C} > 0) \qquad (16)$$

Since C is always positive, equation (11) implies the derivative on v is negative in the first and second quadrants. The derivative is positive in the third and fourth quadrants. Suppose we start at a point in the first quadrant. Then, we have to move along the decreasing direction of v in the graph. This is because the derivative on v is negative (equation (15), i > 0 in first quadrant). This is indicated by the two arrows in figure 9.

In a similar manner, you can derive the dynamic route for the other regions in the graph. Now, you can see why (0,0) is unstable. The arrows close to (0,0) point in the opposite direction. If you start at (0,0), it is practically impossible to stay there. You either move up or down in figure 9. Where you move depends on the noise in the circuit. It is a technical detail and you need not be concerned with it.

Figure 10 shows the other possibilities for stability. I also show a mechanical analogy.

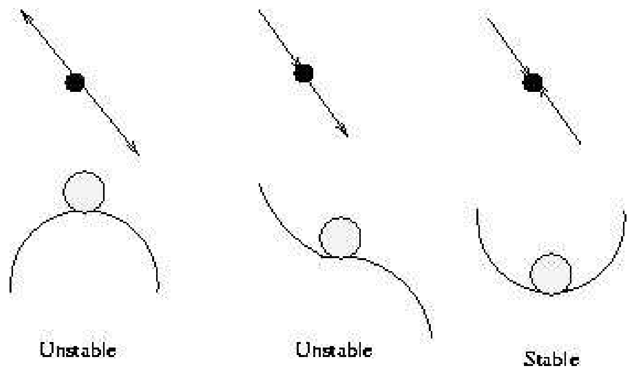


Figure 10. Two instances of instability and one instance of stability. Although the middle example is metastable (unstable in one direction), we still classified as unstable because you do not want your circuit operating at such a point. If noise in your circuit causes the voltages or currents to move in the wrong direction, then you are in trouble! The value will increase without bound and if it is not controlled properly, it may create a dangerous situation.

The other point(s) of interest in the circuit are where two arrows meet – in the second and fourth quadrants in figure 9. Now, the circuit cannot stop once the voltage and current reach the value. The points are not equilibrium points (i does not equal zero). So, what happens? Turns out, one of the variables (current in our case) jumps *instantaneously.* These are indicated by the dotted lines in figure 9, the direction of jump is also shown. These points are also called **impasse points.** Notice how the jump is along values of constant voltage. Physically, this makes sense. Voltage across a capacitor cannot change instantaneously, so the current jumps.

Now, you can see how the circuit oscillates. If we start at (0,0), we will eventually reach one of the impasse points. Then, we have an instantaneous jump. After a while, we will reach another impasse point and the cycle continues.

Intuitively, why does this circuit oscillate? We know the dynamics of the circuit leads to the impasse points. Turns out once the circuit reaches an impasse point, the power supply provides the necessary energy to overcome the impasse point. In other words, you cannot have an oscillator without a power supply. Mechanically, think about a simple pendulum. Physically, you always need to give it "a push" to sustain the oscillation. The oscillator above is also called as a "relaxation oscillator" - the output is not a sine wave, but a square wave.

In the concluding section, let us look at a numeric example. For a change, the circuit is not an oscillator, but the concepts for solving an oscillator problem are exactly the same. You will have more chances to solve oscillator problems in the homework and lab.

5. An Example

Consider the circuit shown in figure P6.20 (a) where N is described by the i-v characteristic shown in figure P6.20 (b).

i. Indicate the dynamic-route. Label all equilibrium points and state whether they are stable or unstable.

ii. Suppose $v_c(0) = 15$ V. Find and sketch $v_c(t)$ and $i_c(t)$ for $t \geq 0$. Indicate all pertinent information on the sketches.

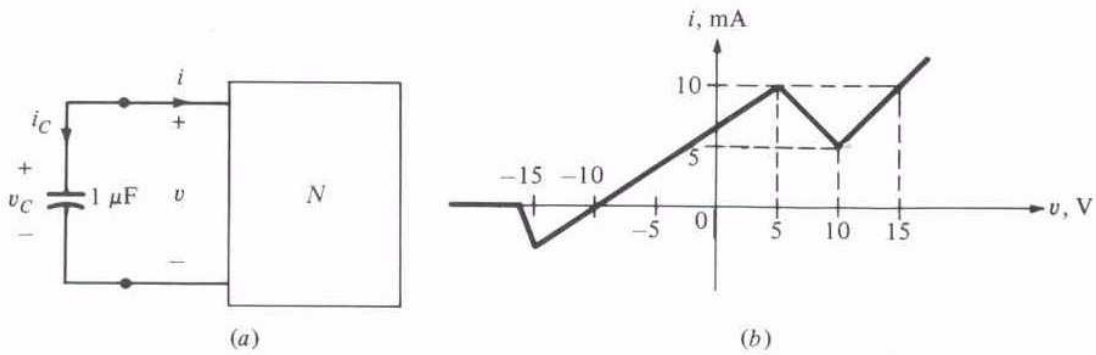


Figure P6.20

$$i_C = C\frac{dv_C}{dt}$$

Now, i_C = -i (by KCL) and v_C = v (by KVL). Therefore:

$$i = -C\frac{dv}{dt}$$

Hence, we have two cases:

$$i < 0 \Rightarrow v' > 0 \quad (1)$$
$$i > 0 \Rightarrow v' < 0 \quad (2)$$

The definition of an equilibrium point is the derivative should be zero, hence:

$$v' = 0 \Rightarrow \frac{-i}{C} = 0 \Rightarrow i = 0 \quad (3)$$

From (1), (2) and (3) we can obtain the dynamic route and equilibrium points shown below: The equilibrium points are (-10,0) and (-15.2,0) (approximate).

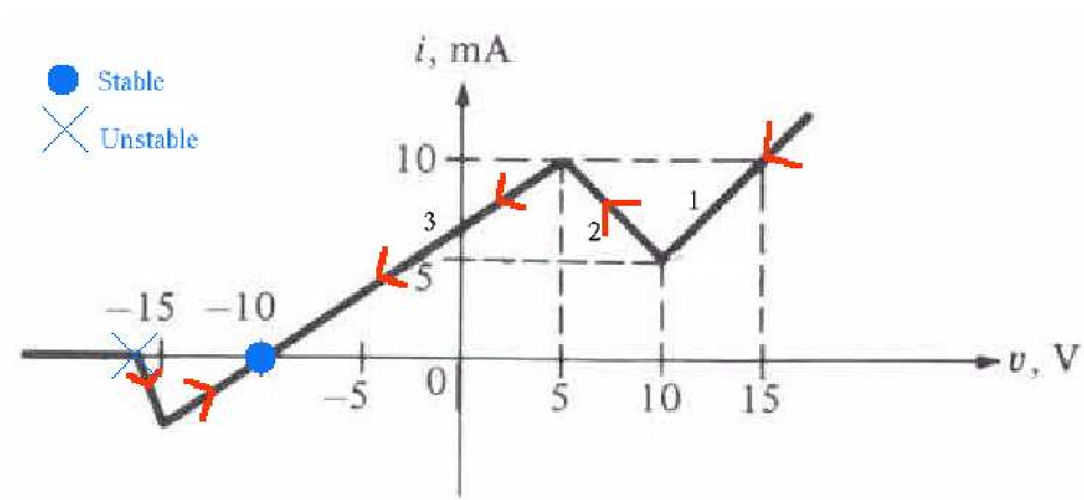


Figure 12: Dynamic route and equilibrium points are shown

It is obvious from the dynamic route that (-10,0) is stable and (-15.2,0) is unstable.

The point (-15.2,0) is a wierd equilibrium point since if we move to the left of the equilibrium point, we get to another equilibrium point. In fact, we have infinitely many equilibrium points to the left of (-15.2,0). You don't have to worry about these details - (-15.2,0) is unstable because we move away from the point in any one direction, we don't get back to the point.

ii. Let us sketch $i_c(t)$ first. However the given i-v graph shows i(t). The easiest way to sketch i_c is to find i(t) first and then use the fact: i_c = -i. Since the system is first-order:

$$i(t) = i_{final} - (i_{final} - i_{initial})e^{\frac{-t}{\tau}}$$

Let us split the i-v graph into three regions depending on the slope:

- In region 1 in figure 12, we have:

$$\tau_1 = RC = \frac{\triangle v}{\triangle i}C = (1k)(1\mu) = 1ms$$

  $i_{initial}$ = 10 mA, since $v_c(0)$ = 15 V. To find i_{final}, we think what would happen if the i-v graph did not change slope at v_c = 10 V. We would see that i(t) would continue decreasing towards 0 A. Therefore, i_{final} = 0 A. Hence:

$$i_1(t) = 10e^{\frac{-t}{1ms}}$$

  $i_1(t)$ is in mA. Obviously, this equation is valid only till $i(t_1)$ = 5 mA. Hence,

$$5 = 10e^{\frac{-t_1}{1ms}}$$
$$t_1 = -ln(0.5)(1ms)$$
$$t_1 = 0.69ms$$

  Therefore:

$$i_1(t) = 10e^{\frac{-t}{1ms}}, 0 \leq t < 0.69ms$$

- In region 2, τ IS GOING TO BE NEGATIVE. Therefore, our system is unstable. However, we can calculate i_{final} by extrapolating the graph "backwards in time". This is because our first-order equation becomes:

$$i(t) = i_{final} - (i_{final} - i_{initial})e^{\frac{t}{\tau}}$$

and i(t) $\rightarrow$ i_{final} as t $\rightarrow$ -∞. Therefore, by extrapolating line 2 backwards, i_{final} = 0 A. Obviously, $i_{initial}$ = 5 mA.

$$i_2(t) = i_{final} - (i_{final} - i_{initial})e^{\frac{(t-0.69ms)}{1ms}} = 5e^{\frac{(t-0.69ms)}{1ms}}$$

Notice that we need to shift our current function by 0.69 ms; because this function starts only at t = 0.69 ms. The current again changes slope when $i_2(t_2)$ = 10 mA. Hence:

$$10 = 5e^{\frac{(t_2-0.69ms)}{1ms}}$$
$$t_2 = ln(2)(1ms) + 0.69ms$$
$$t_2 = 1.38ms$$

Thus, i(t) changes slope when t = 1.38 ms. This value intuitively makes sense: the i-v graph in region 2 is symmetrical to region 1. Hence:

$$i_2(t) = 5e^{\frac{(t-0.69ms)}{1ms}}, 0.69ms \leq t < 1.38ms$$

- Calculation of i_3(t) is similar to i_1(t) and i_2(t). Notice the time constant here is different:

$$\tau_3 = RC = \frac{\triangle v}{\triangle i}C = (1.5k)(1\mu) = 1.5ms$$

$i_{initial}$ = 10 mA, i_{final} = 0 mA. The equation is:

$$i_3(t) = 10e^{\frac{-(t-1.38ms)}{1.5ms}}, t \geq 1.38ms$$

Are we done for i(t)? Yes, but the question asked for i_c(t) = -i(t). Finally:

$$\begin{aligned} i_c(t) &= -10e^{\frac{-t}{1ms}}, 0 \leq t < 0.69ms \\ &= -5e^{\frac{(t-0.69ms)}{1ms}}, 0.69ms \leq t < 1.38ms \\ &= -10e^{\frac{-(t-1.38ms)}{1.5ms}}, t \geq 1.38ms \end{aligned}$$

i_c(t) is in mA above.

A plot of i_c(t) versus t is shown below.

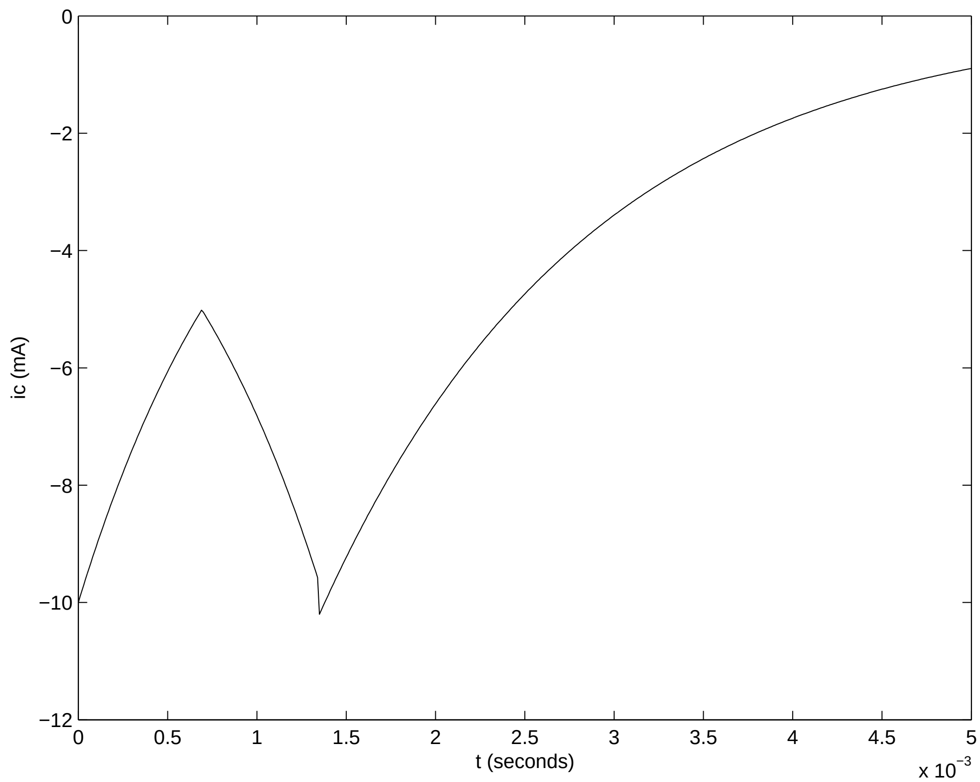


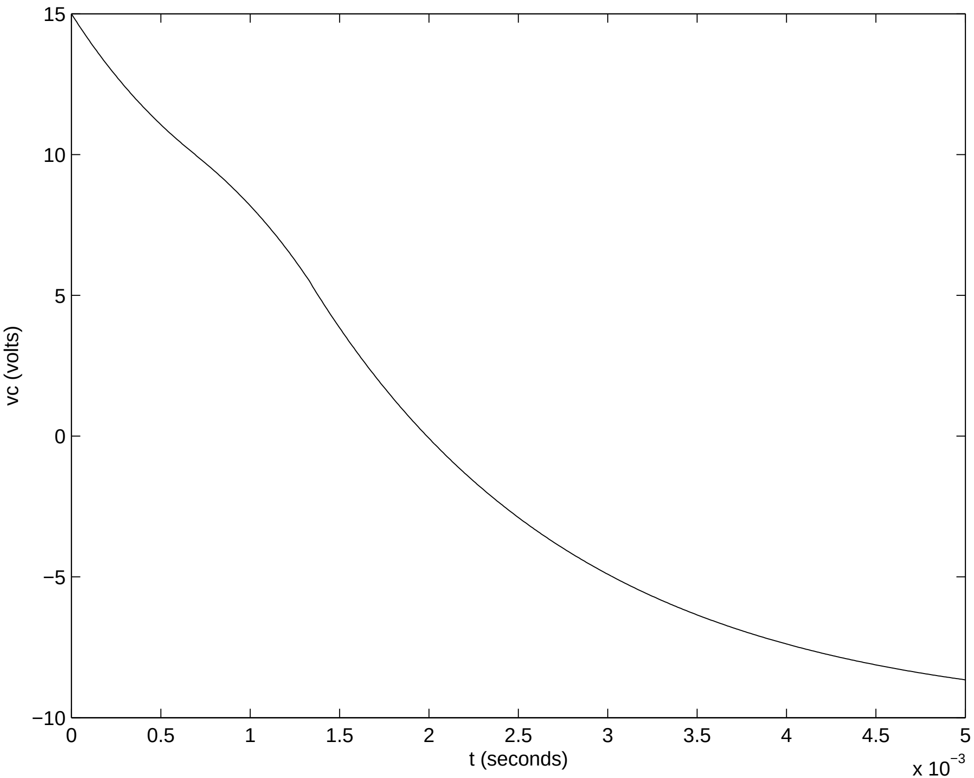
Figure 13: i_c versus t. Ignore the little "kink" at t = 1.38 ms, that is due to MatLAB approximation

v_c(t) can be found in the same way as i_c(t). I am just giving the final answer and plot. If you need to know how to do this in detail, feel free to stop by a TA's office hours.

$$
\begin{aligned}
v_c(t) &= 5 + 10e^{\frac{-t}{1ms}}, 0 \leq t < 0.69ms \\
&= 15 - 5e^{\frac{(t-0.69ms)}{1ms}}, 0.69ms \leq t < 1.38ms \\
&= -10 + 15e^{\frac{-(t-1.38ms)}{1.5ms}}, t \geq 1.38ms
\end{aligned}
$$

v_c(t) is in volts above.

A plot of v_c(t) versus t is shown below.



15
10
5
0
−5
−10
vc (volts)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t (seconds)
x 10−3

6. References

1. *Linear and Nonlinear Circuits, chapters 4 and 5.* Chua, Leon O., Desoer and Kuh. McGraw-Hill. ISBN# 0-07-010898-6